On Non-Relativistic Conformal Field Theory and Trapped Atoms: Virial Theorems and the State-Operator Correspondence in Three Dimensions

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Abstract

The field theory of nonrelativistic fermions interacting via contact interactions can be used to calculate the properties of few-body systems of cold atoms confined in harmonic traps. The state-operator correspondence of Non-Relativistic Conformal Field Theory (NRCFT) shows that the energy eigenvalues (in oscillator units) of N harmonically trapped fermions can be calculated from the scaling dimensions of N-fermion operators in the NRCFT. They are also in one-to-one correspondence with zero-energy, scale-invariant solutions to the N-body problem in free space. We show that these two mappings of the trapped fermion problem to free space problems are related by an automorphism of the $SL(2,R)$ algebra of the conformal symmetry of fermions at the unitary limit. This automorphism exchanges the internal Hamiltonian of the gas with the trapping potential and hence provides a novel method for deriving virial theorems for trapped Fermi gases at the unitary limit. We also show that the state-operator correspondence can be applied directly in three spatial dimensions by calculating the scaling dimensions of two- and three-fermion operators and finding agreement with known exact results for energy levels of two and three trapped fermions at the unitary limit.

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I. INTRODUCTION

The problem of few-body atomic interactions in the presence of external confining potentials is motivated by recent advances in experimental atomic physics as well as theory. Recent experiments have realized optical lattices with two confined in a potential well [1, 2, 3, 4]. Such atomic states have been proposed for implementing quantum logic gates [5, 6, 7, 8]. Theoretically, the problem of two atoms interacting via short-range forces has been solved in Ref. [9], see also Refs.[10, 11, 12, 13, 14, 15, 16]. An experimental confirmation of the prediction for the ground state energy of two trapped atoms as a function of scattering length was recently performed in Ref. [1]. If the scattering length of the atoms is tuned to infinity and effective range terms are neglected then the S-wave scattering cross section is $4\pi/p^2$ where p is the relative momentum. This cross section is at the upper limit allowed by unitarity. The quantum mechanics problem of three particles at the unitary limit in the presence of a harmonic potential has been solved in Ref. [17], see also Ref. [18].

An outstanding open problem is the many-body problem of fermions at unitary, which has been investigated by numerous authors using a wide variety of methods. Gases of trapped fermions whose interactions have been tuned to the unitary limit by means of a Feshbach resonance have been realized experimentally [19, 20, 21]. For a review of experimental and theoretical results, see Refs. [22, 23]. Both the homogeneous unitary Fermi gas as well as the unitary Fermi gas in the presence of harmonic traps are clearly of interest. Though the many-body physics problem presents physical challenges not present in the two- and three-body problems, the existence of exact solutions for $N = 2$ and 3, where N is the number of fermions, can provide important inputs for the case of arbitrary N. For example, Ref. [24] proposes a scale-invariant density functional for the unitary Fermi gas whose parameters are fixed by matching the known analytic solutions for two fermions at the unitary limit in the harmonic trap. Corrections due to a finite scattering length and effective range are included in Ref. [25]. This provides another motivation for studying few-body trapped fermion problems.

An interesting theoretical development is the state-operator correspondence which relates the problem of finding the energy eigenvalues of N trapped fermions at the unitary limit to the problem of finding the scaling dimensions of primary operators in a Non-Relativistic Conformal Field Theory (NRCFT) [26]. Note that the NRCFT is defined in the absence of an external potential, so the state-operator correspondence relates a property of the theory of N fermions in free space to the properties of N trapped fermions. Another mapping of the trapped N-fermion problem to the free space N-fermion problem is derived in Ref. [27]. These authors map the problem of harmonically trapped fermions at the unitary limit to the problem of finding zero-energy, scale invariant eigenfunctions of the N-body problem in the absence of any external potential. One goal of this paper is to better understand the relationship between these two mappings.

The other main goal of this paper is to show how the state-operator correspondence can be applied directly in three dimensions. For two spatial dimensions ($d = 2$), the theory of fermions at the unitary limit is equivalent to noninteracting fermions while in $d = 4$ the theory is equivalent to noninteracting bosons [28]. Therefore, in $2 + \epsilon$ dimensions and $4 - \epsilon$ dimensions a perturbation theory in ϵ can be used to analyze the properties of unitary fermions [29, 30]. In Ref. [26], the ϵ expansion is combined with the state-operator correspondence to calculate the energy levels of few-body atomic systems in harmonic traps. Operator scaling dimensions are calculated in a perturbative series in ϵ and Padé approx-

imants are used to interpolate between $d = 2$ and $d = 4$ to obtain results for the most physically interesting case of $d = 3$. In this paper, we will illustrate how the state-operator correspondence can be applied directly in $d = 3$. Using a low energy effective field theory for two component fermions interacting via S-wave contact interactions, we calculate the scaling dimensions of S-wave $N = 2$ and $N = 3$ fermion operators and find agreement with the exact solutions for the energy levels of two trapped fermions, as well as the lowest energy state of three trapped fermions in an S-wave.

The low energy interactions of few-body systems can be studied using the methods of effective field theory. These methods are useful when the typical momentum times the range of the interactions is much less than one. This is the case for cold atoms, where a complete model for the potential is not required and many quantities can be computed in terms of the S-wave scattering length alone. At these energies the S-wave scattering amplitude for two fermions with momentum $\pm p$ is

$$\begin{aligned}\mathcal{A} &= \frac{4\pi}{M}\frac{1}{p\cot\delta(p) - ip} \\ &= \frac{4\pi}{M}\frac{1}{-1/a + r_0\, p^2/2 + \ldots - ip}\,,\end{aligned}$$

where a is the scattering length and r_0 is the effective range. In the limit $r_0\, p \ll 1$, effective range corrections can be neglected and the two particle scattering amplitude is exactly reproduced by a nonrelativistic field theory with a single S-wave contact interaction. The Lagrangian for this nonrelativistic field theory is

$$\mathcal{L} = \psi^\dagger\left(i\partial_t + \frac{\nabla^2}{2M}\right)\psi - \frac{C_0(\mu)}{4}\psi^\dagger\psi^\dagger\psi\psi\,, \tag{1}$$

where ψ is a two-component field operator that annihilates fermion quanta. Here $\psi\psi = \epsilon^{\alpha\beta}\psi_\alpha\psi_\beta$, so scattering occurs in the S-wave, spin-singlet channel only. The coupling constant, $C_0(\mu)$, is given by

$$C_0(\mu) = \frac{4\pi}{M}\frac{1}{-\mu + 1/a}\,, \tag{2}$$

where μ is the dimensional regularization (DR) parameter and Power Divergence Subtraction (PDS) scheme is used to regulate loop integrals [31]. In DR, loop integrals which are linearly divergent when regulated with a cutoff can become finite because DR discards power law divergences. These can be restored within the framework of DR by subtracting poles in one lower dimension, then the DR parameter, μ, enters the calculation of loop integrals in the same way that a hard cutoff would. In the Minimal Subtraction (MS) scheme, where the linear divergences are discarded, $C_0 = 4\pi a/M$. Thus, it is clear that to make sense of the Lagrangian in the limit $a \to \pm\infty$, one needs to use a hard cutoff, PDS, or some other regularization scheme that keeps track of linear divergences. We will see below that in order to obtain scaling dimension of operators that are consistent with the state-operator correspondence of NRCFT, we must also use one of these schemes.

Since the work of Ref. [32], it is known that the $a \to \pm\infty$ limit of the theory of two-component fermions in Eq. (1) is conformally invariant. [1] The nonrelativistic scale transfor-

[1] For bosons or fermions with more than two degrees of freedom, an S-wave three-body contact interaction is relevant and violates scale invariance [33].

mation is

$$\vec{x}' = \lambda\,\vec{x} \qquad t' = \lambda^2\,t \qquad \psi'(\vec{x}\,',t') = \lambda^{-3/2}\psi(\vec{x},t)\,, \tag{3}$$

and the nonrelativistic conformal transformation is

$$\vec{x}' = \frac{\vec{x}}{1+c\,t} \qquad t' = \frac{t}{1+c\,t} \qquad \psi'(\vec{x}\,',t') = (1+c\,t)^{3/2}\exp\left(\frac{-iM\,c\,\vec{x}^{\,2}}{2(1+c\,t)}\right)\psi\,(\vec{x},t)\;. \tag{4}$$

It is straightforward to show that the scale and conformal transformations are symmetries of the non-interacting theory. For generic values of a the contact interaction in Eq. (1) breaks these symmetries. Since the two-particle S-wave cross section is independent of any scale when $a \to \pm\infty$, it is natural to expect the theory to be scale and conformally invariant in this limit. Ref. [32] showed that the off-shell $2 \to 2$ scattering amplitude calculated in the theory of Eq. (1) is invariant under the Ward identities implied by scale and conformal transformations when $a \to \pm\infty$. Ref. [26] gives a simple argument for why any particle number conserving theory that is scale invariant should also be invariant under conformal transformations.

When $a \to \pm\infty$, the Hamiltonian, H, the generator of nonrelativistic scale transformations, D, and the generator of conformal transformations, C, form an $SL(2,R)$ algebra. We will show below that the two mappings of the trapped fermion problem to free space fermion problems that were discussed earlier are related by an automorphism of the $SL(2,R)$ group. This automorphism exchanges the generators C and H. Since the generator C is just the external potential for the trapped fermions, this automorphism interchanges the trapping potential and the internal Hamiltonian of the gas. Therefore, the automorphism can be used to provide a novel group theoretical derivation of virial theorems for trapped Fermi gases at the unitary limit. A virial theorem was first derived using the assumption of universality, the local density approximation, and thermodynamic arguments in Ref. [34]. The virial theorem was then rederived and generalized using the wavefunctions of the pseudopotential model of the unitary Fermi gas in Ref. [27]. Another derivation of the virial theorem using the Hellmann-Feynman theorem appears in Ref. [35]. Our derivation is novel in that the approach is group theoretical and relies only on the $SL(2,R)$ algebra. The virial theorems hold for N-body energy eigenstates as well for thermal ensembles, and can be applied to spin polarized or unpolarized gases.

The paper is organized as follows: in the next section, we review basic facts about NRCFT's and the state-operator correspondence, as well as the correspondence of Ref. [27] which relates eigenstates of trapped fermions to zero-energy, scale-invariant eigenfunctions in free space. We discuss the automorphism of the $SL(2,R)$ algebra which relates these mappings and show how it can be used to derive the virial theorems. In section III, we derive the scaling dimension of operators with $N = 2$ and $N = 3$ and show that these agree with analytic results for the energies of trapped fermions. In Section IV, we conclude. In the Appendix, we solve the problem of two trapped atoms with arbitrary short-range interactions. This was first done for arbitrary scattering length in Ref. [9] using the method of pseudopotentials. Here we solve the problem by calculating Green's functions for two particles in the trap using the field theory of Eq. (1).

II. NRCFT, $SL(2,R)$ AUTOMORPHISMS, AND VIRIAL THEOREMS

In this section we begin by reviewing NRCFT and the two mappings of the problem of trapped fermions at the unitary limit to free space problems [26, 27]. We then show that these two mappings are related by an automorphism of the $SL(2,R)$ conformal symmetry algebra. This is the main result of this section. The automorphism is realized by a unitary transformation, generated by the Hamiltonian of the trapped fermions, that interchanges the internal Hamiltonian of the Fermi gas with the external trapping potential. This unitary transformation can then be used to provide a simple derivation of the virial theorems for trapped fermions at the unitary limit.

The many-body Hamiltonian for harmonically trapped fermions in second quantized form is the sum of an internal Hamiltonian, $H_{\rm int}$, and an external potential, $V_{\rm ext}$, which are given by

$$\begin{aligned} H_{\rm int} &= \int d^3x \left[\psi^\dagger \left(-\frac{\nabla^2}{2M}\right)\psi + \frac{C_0(\mu)}{4}\,\psi^\dagger\psi^\dagger\psi\psi\right] \\ V_{\rm ext} &= \int d^3x\,\frac{1}{2}M\,\omega^2\,\vec{x}^{\,2}\,\psi^\dagger\psi\,. \end{aligned} \tag{5}$$

After the following rescaling,

$$\vec{x} \to \frac{\vec{x}}{\sqrt{M\omega}} \qquad \psi \to (M\omega)^{3/4}\psi \qquad \mu \to \sqrt{M\omega}\,\mu \qquad a \to \frac{a}{\sqrt{M\omega}}\,, \tag{6}$$

which renders all these quantities dimensionless (we are using $\hbar = 1$ units), we find

$$\begin{aligned} H_{\rm int} &= \omega \int d^3x \left(\psi^\dagger \left(-\frac{\nabla^2}{2}\right)\psi + \frac{\hat{C}_0(\mu)}{4}\,\psi^\dagger\psi^\dagger\psi\psi\right) \\ &\equiv \omega\, H \\ V_{\rm ext} &= \omega \int d^3x\,\frac{1}{2}\,\vec{x}^{\,2}\,\psi^\dagger\psi \\ &\equiv \omega\, C\,. \end{aligned} \tag{7}$$

Here we have defined $\hat{C}_0(\mu) = M\,C_0(\mu)$, so that $\hat{C}_0(\mu)$ is independent of M. This shows that we can set $M = \omega = 1$ and measure all energies in units of the fundamental oscillator energy, ω. Lengths are measured in units of $a_{\rm osc} = 1/\sqrt{M\omega}$. In this section, we will use these units and work with H and C rather than $H_{\rm int}$ and $V_{\rm ext}$. C is the generator of conformal transformations [26].

If we modify the definition of the scale transformation to include the appropriate transformation on μ,

$$\vec{x}\,' = \lambda\,\vec{x} \qquad t' = \lambda^2\,t \qquad \mu' = \lambda^{-1}\mu \qquad \psi'(\vec{x}\,', t') = \lambda^{-3/2}\,\psi(\vec{x}, t)\,, \tag{8}$$

we find

$$H' = \lambda^{-2} H\,, \tag{9}$$

when $a = \pm\infty$. Though the scale transformation of Eq. (8) differs from that of Eq. (3) by additional rescaling of μ, the Ward identities derived in Ref. [32] will still hold for any renormalized Green's function that is μ independent. Likewise, to see the conformal invariance

of the theory defined by Eq. (1) explicitly, one must modify the conformal transformation in Eq. (4) to include a time-dependent rescaling of μ. If D is the generator of scale transformations then

$$H' = e^{i\alpha D} H e^{-i\alpha D} = e^{-2\alpha} H \qquad (\alpha = \log \lambda)\,, \tag{10}$$

which gives the commutation relation $[D, H] = 2iH$.

In a NRCFT, the Hamiltonian, H, dilatation operator, D, and conformal generator, C, obey the following commutation relations:

$$\begin{aligned} &a) \qquad [D, H] = 2iH \\ &b) \qquad [D, C] = -2iC \\ &c) \qquad [H, C] = -iD\,, \end{aligned} \tag{11}$$

which are the commutation relations of the group $SL(2, R)$. For the theory of Eq. (1), we have given H and C above and D is given by[2]

$$D \;=\; \int d^3x\, \vec{x} \cdot \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\nabla} \right) \psi = \int d^3x\, \vec{x} \cdot \vec{j}(x)\,. \tag{12}$$

where $\vec{j}(x)$ is the particle current density. Eq. (11b) follows automatically from the definitions of D and C and the equal time commutation relations of ψ and $\psi^\dagger$. Eq. (11c) is actually quite general and will hold for any theory in which particle number is locally conserved. Note that $C = \int d^3x\, \frac{1}{2} \vec{x}^{\,2}\, n(x)$, where $n(x)$ is the particle density operator. The commutator of the Hamiltonian is proportional to the divergence of the particle current [26]

$$[H, n(x)] = -i\partial_t n(x) = i\vec{\nabla} \cdot \vec{j}(x)\,, \tag{13}$$

due to current conservation. Eq. (11b) follows by multiplying Eq. (13) by $\vec{x}^{\,2}/2$ and integrating over all space. So if Eq. (11a), which is the requirement of scale invariance, is satisfied in a particle number conserving theory then the theory will also be invariant under the full $SL(2, R)$ conformal group. To complete the algebra of the Schrödinger group (the largest space-time symmetry group of free nonrelativistic quantum mechanics), we also need the commutation relations of H, D, and C with other symmetry generators: momentum, $\vec{P}$, angular momentum, $\vec{J}$, Galilean boosts, $\vec{K}$, and particle number, N. The nonvanishing commutators involving K_i, P_i, D, and N are:

$$[K_i, P_j] = i\, N\delta_{ij} \qquad [D, P_i] = iP_i \qquad [D, K_j] = -iK_i\,. \tag{14}$$

Commutation relations involving $\vec{J}$ are easily deduced from rotational invariance.

Primary operators in the NRCFT are defined by $\mathcal{O} \equiv \mathcal{O}(\vec{x} = 0, t = 0)$ and

$$[\vec{K}, \mathcal{O}] = [C, \mathcal{O}] = 0. \tag{15}$$

[2] The explicit expressions for C and D are valid at $t = 0$. For arbitrary t, $C(t) = C(0) + t^2 H - tD(0)$ and $D(t) = D(0) - 2tH$. The explicit time dependence can be fixed by requiring $\dot{A}(t) = -i[A, H] + \partial A/\partial = 0, A = C, D$, which is required for conserved charges that generate a symmetry of the Hamiltonian. See Ref. [36] for a one-dimensional conformally invariant quantum mechanical system.

Note that primary operators are defined to be located at the origin of space and time. The particle number ($N_\mathcal{O}$) and scaling dimension ($\Delta_\mathcal{O}$) of the primary operator are defined by

$$\begin{aligned} [D,\mathcal{O}] &= i\Delta_\mathcal{O}\mathcal{O}\,, \\ [N,\mathcal{O}] &= N_\mathcal{O}\mathcal{O}\,. \end{aligned} \tag{16}$$

If we translate the primary operator $\mathcal{O}$ to another point in space-time

$$\mathcal{O}(\vec{x},t) = e^{iHt-i\vec{P}\cdot\vec{x}}\mathcal{O}(0)e^{-iHt+i\vec{P}\cdot\vec{x}}\,, \tag{17}$$

then it is straightforward to show using the commutation relations listed above that

$$\begin{aligned} [\vec{K},\mathcal{O}] &= (-it\partial_i + N_\mathcal{O}x_i)\mathcal{O} \\ [C,\mathcal{O}] &= -i(t^2\partial_t + t\vec{x}\cdot\vec{\partial} + t\Delta_\mathcal{O})\mathcal{O} + \frac{\vec{x}^{\,2}}{2}N_\mathcal{O}\mathcal{O}. \end{aligned} \tag{18}$$

The field ψ has $N_\psi = -1$ and $\Delta_\psi = d/2$, where d is the dimensionality of space. The density operator, $\psi^\dagger\psi$, has $N_{\psi^\dagger\psi} = 0$ and $\Delta_{\psi^\dagger\psi} = d$. For a finite conformal transformation we have

$$\begin{aligned} \mathcal{O}'(\vec{x},t) &= e^{-i\lambda C}\mathcal{O}(\vec{x},t)e^{i\lambda C} \\ &= \frac{1}{(1+\lambda\,t)^{\Delta_\mathcal{O}}}\exp\left(\frac{-iN_\mathcal{O}\,\lambda\,\vec{x}^{\,2}}{2(1+\lambda\,t)}\right)\mathcal{O}\left(\frac{\vec{x}}{1+\lambda\,t},\frac{t}{1+\lambda\,t}\right). \end{aligned} \tag{19}$$

which agrees with Eq. (4) for the case $\mathcal{O} = \psi$.

Next we discuss consequences following from the algebra in Eq. (11). Let us define

$$\begin{aligned} H_{\rm osc} &\equiv H + C \\ L_\pm &= H - C \pm iD\,. \end{aligned} \tag{20}$$

The $L_\pm$ are ladder operators that raise and lower energy eigenvalues of $H_{\rm osc}$ by two oscillator units, as can be seen from the commutation relations

$$\begin{aligned} [L_\pm, H_{\rm osc}] &= \mp 2L_\pm \\ [L_-, L_+] &= 4H_{\rm osc}\,, \end{aligned} \tag{21}$$

which are easily derived from Eq. (11). Eigenstates of $H_{\rm osc}$ come in infinite towers of equally spaced states. The ground state of one of these towers is denoted by $|\psi_0\rangle$ which satisfies $L_-|\psi_0\rangle = 0$.

The problem of finding energy eigenstates for the trapped particles can be mapped to the free space theory in one of two ways. The method of Ref. [26] begins by noting that

$$e^H L_- e^{-H} = -C\,, \tag{22}$$

and furthermore

$$\begin{aligned} C\mathcal{O}^\dagger|0\rangle &= [C,\mathcal{O}^\dagger]|0\rangle \\ &= 0\,, \end{aligned} \tag{23}$$

where $\mathcal{O}$ is a primary operator and $|0\rangle$ is the vacuum. Then

$$L_- e^{-H}\mathcal{O}^\dagger|0\rangle = -e^{-H}C\mathcal{O}^\dagger|0\rangle = 0 \tag{24}$$

so the ground state of the tower is $|\psi_0\rangle = e^{-H}\mathcal{O}^\dagger|0\rangle$. We find that from any primary operator we can construct a tower of eigenstates of $H_{\rm osc}$. It is straightforward to show that [26]

$$H_{\rm osc}e^{-H}\mathcal{O}^\dagger|0\rangle = e^{-H}(C - iD)\mathcal{O}^\dagger|0\rangle = \Delta_{\mathcal{O}}e^{-H}\mathcal{O}^\dagger|0\rangle\,. \tag{25}$$

Thus the scaling dimension of the operator in the NRCFT (in the absence of an external potential) gives the ground state energy (in oscillator units) of the corresponding state $|\psi_0\rangle$ in the problem with an external harmonic potential.

Ref. [27] pointed out another correspondence between eigenstates of the trapped fermions, $H_{\rm osc}$, and zero-energy, scale invariant eigenstates of H. The result of Ref. [27] can be obtained starting with a relation analogous to Eq. (22),

$$e^C L_- e^{-C} = H\,. \tag{26}$$

From this relation it is clear that $|\psi_0\rangle = e^{-C}|\psi_\nu\rangle$, where $|\psi_\nu\rangle$ is a zero-energy eigenstate of the Hamiltonian, $H|\psi_\nu\rangle = 0$. For this state to be an eigenstate of $H_{\rm osc}$, it must be an eigenstate of iD as well,

$$iD|\psi_\nu\rangle = \left(\nu + \frac{3}{2}N\right)|\psi_\nu\rangle\,. \tag{27}$$

The energy eigenvalue of $|\psi_0\rangle$ is

$$\begin{aligned} H_{\rm osc}|\psi_0\rangle &= e^{-C}\,e^{C}H_{\rm osc}\,e^{-C}|\psi_\nu\rangle \\ &= e^{-C}(H + iD)|\psi_\nu\rangle \\ &= \left(\nu + \frac{3}{2}N\right)|\psi_0\rangle\,. \end{aligned} \tag{28}$$

To understand the significance of ν, note that the N-body wavefunction associated with the state $|\psi_\nu\rangle$ is

$$\psi_\nu(\vec{x}_i) = \langle 0|\prod_{i=1}^{N}\psi(\vec{x}_i)|\psi_\nu\rangle\,. \tag{29}$$

Using

$$e^{-i\alpha D}\psi(\vec{x}_i)e^{i\alpha D} = e^{\frac{3}{2}\alpha}\psi(e^\alpha\vec{x}_i) \tag{30}$$

it is easily seen that

$$\psi_\nu\left(\frac{\vec{x}_i}{\Lambda}\right) = \Lambda^{-\nu}\psi_\nu(\vec{x}_i)\,, \tag{31}$$

so the N-body wavefunction for the state $|\psi_\nu\rangle$ is a homogeneous function of the N-body coordinates. Note that the N-body wavefunction for the trapped problem is given by

$$\begin{aligned} \langle\vec{x}_i|\psi_0\rangle &= \langle\vec{x}_i|e^{-C}|\psi_\nu\rangle \\ &= e^{-\sum_i \vec{x}_i^2/2}\psi_\nu(\vec{x}_i)\,. \end{aligned} \tag{32}$$

To understand the relationship between the two mappings of the trapped problem to free space problems, we observe that Eqs. (22) and (26) are related by the following automorphism of the $SL(2,R)$ algebra

$$H \leftrightarrow C\,, \qquad D \to -D\,. \tag{33}$$

This is an automorphism of the $SL(2,R)$ algebra which is implemented by a similarity transformation using the elements

$$g_n = e^{i\pi(n+1/2)H_{\rm osc}}\,, \tag{34}$$

whose action on the generators of $SL(2,R)$ is [3]

$$g_n \begin{pmatrix} H \\ C \\ D \end{pmatrix} g_n^{-1} = \begin{pmatrix} C \\ H \\ -D \end{pmatrix}\,. \tag{36}$$

These identities immediately lead to the virial theorems for trapped fermions at the unitary limit derived in Refs. [27, 34]. The thermal expectation value of an arbitrary operator, $\hat{\mathcal{O}}$, is given by

$$\langle \hat{O} \rangle = {\rm Tr}[e^{-\beta(H_{\rm osc}-\mu_+ N_+ - \mu_- N_-)}\hat{O}]\,, \tag{37}$$

where we have included separate chemical potentials, μ_+ and μ_-, for spin up and spin down fermions, respectively, so our results can be applied to spin polarized as well as unpolarized gases. For the expectation value in Eq. (37), or the expectation $\hat{O}$ in an eigenstate of $H_{\rm osc}$, we have $\langle g_n \hat{O} g_n^{-1}\rangle = \langle \hat{O}\rangle$, because $[H_{\rm osc}, N_\pm] = 0$. Therefore,

$$\langle H^n \rangle = \langle C^n \rangle\,, \qquad \langle D^{2n+1}\rangle = 0\,. \tag{38}$$

For $n = 1$, this implies $\langle H_{\rm osc}\rangle = E_0 = \langle H + C\rangle = 2\langle C\rangle$ which is the virial theorem first derived in Ref. [34]. The generalization to arbitrary moments of C in the ground state is straightforward:

$$\begin{aligned}
\langle\psi_0|C^n|\psi_0\rangle &= \langle\psi_0|C^{n-1}\,C|\psi_0\rangle \\
&= \langle\psi_0|C^{n-1}\left(\frac{H_{\rm osc}}{2} - \frac{L_+ + L_-}{4}\right)|\psi_0\rangle \\
&= \frac{E_0}{2}\langle\psi_0|C^{n-1}|\psi_0\rangle + \frac{1}{4}\langle\psi_0|[L_+, C^{n-1}]|\psi_0\rangle \\
&= \frac{E_0}{2}\langle\psi_0|C^{n-1}|\psi_0\rangle + \frac{1}{4}\langle\psi_0|2(n-1)\,C^{n-1} + [H, C^{n-1}]|\psi_0\rangle \\
&= \frac{E_0}{2}\langle\psi_0|C^{n-1}|\psi_0\rangle + \frac{1}{4}\langle\psi_0|2(n-1)\,C^{n-1} + [H_{\rm osc}, C^{n-1}]|\psi_0\rangle \\
&= \frac{E_0 + (n-1)}{2}\langle\psi_0|C^{n-1}|\psi_0\rangle\,.
\end{aligned} \tag{39}$$

[3] Eq. (36) is a special case of

$$e^{i\theta H_{\rm osc}} \begin{pmatrix} H \\ C \\ D \end{pmatrix} e^{-i\theta H_{\rm osc}} = \begin{pmatrix} \cos^2\theta & \sin^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta \\ \sin 2\theta & -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} H \\ C \\ D \end{pmatrix} \tag{35}$$

The automorphism in Eq. (36) is obtained for $\sin\theta = \pm 1$.

We have used $L_-|\psi_0\rangle = 0 = \langle\psi_0|L_+$. This simple recursion relation immediately gives all higher moments of the trapping potential which can be written in closed form as [27]

$$\langle C^n \rangle = \frac{\Gamma[E_0 + n]}{2^n \Gamma[E_0]} \,. \tag{40}$$

This concludes our general discussion of NRCFT. The main result of this section is the automorphism of $SL(2,R)$ which relates the two known mappings of the trapped N-fermion problem to problems involving the N fermions in free space. This automorphism provides a simple, group theoretical method for deriving virial theorems for both eigenstates and for thermal expectation values with arbitrary chemical potential for the two spin components. In the next section of the paper, we will show that the state-operator correspondence can be used directly in $d = 3$ using the effective field theory of Eq. (1).

III. STATE-OPERATOR CORRESPONDENCE IN $d = 3$

A. Two fermions

The problem of two fermions interacting via short-range interactions in the presence of an external harmonic potential is exactly solvable [9]. This solution is reviewed in the Appendix. The ground state energy of two fermions at the unitary limit in a harmonic trap is 2, in oscillator units. In this section, we verify the state-operator correspondence by evaluating the anomalous dimension of the composite operator $\psi\psi$ using the NRCFT of Ref. [32]. We compute the matrix element

$$\langle 0|Z_{\psi\psi}(\mu)\psi\psi|\vec{p}, -\vec{p}\,\rangle \,, \tag{41}$$

which is given by the Feynman diagrams in Fig. 1. The factor $Z_{\psi\psi}(\mu)$ is required for composite operator renormalization. We work in the center of mass frame, where E is the total kinetic energy and the momentum of each particle is $p = |\vec{p}\,| = \sqrt{ME}$. The diagrams form a geometric series

$$\langle 0|Z_{\psi\psi}(\mu)\psi\psi|\vec{p}, -\vec{p}\,\rangle \;=\; \frac{Z_{\psi\psi}(\mu)}{1 - C_0(\mu) G^0_E(\vec{0},\vec{0})} \,, \tag{42}$$

where

$$\begin{aligned} G^0_E(\vec{0},\vec{0}) \;&=\; \left(\frac{\mu}{2}\right)^{3-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{E - \vec{l}^{\,2}/M} \\ &=\; -\frac{M}{4\pi}\left(\mu - \sqrt{-p^2 + i\epsilon}\right) \,. \end{aligned} \tag{43}$$

The first line of Eq. (43) is obtained after evaluating by contour integration the energy integral in the one-loop bubble graph that is pictured in Fig. 1. Note that the one-loop graph in the NRCFT is related to Green's function for the free two-body Hamiltonian:

$$\begin{aligned} G^0_E(\vec{x},\vec{y}) \;&=\; \langle \vec{x}| \frac{1}{E - H_0} |\vec{y}\rangle \\ &=\; \left(\frac{\mu}{2}\right)^{3-d} \int \frac{d^d l}{(2\pi)^d} \frac{e^{i\vec{l}\cdot(\vec{x}-\vec{y})}}{E - \vec{l}^{\,2}/M} \,, \end{aligned} \tag{44}$$

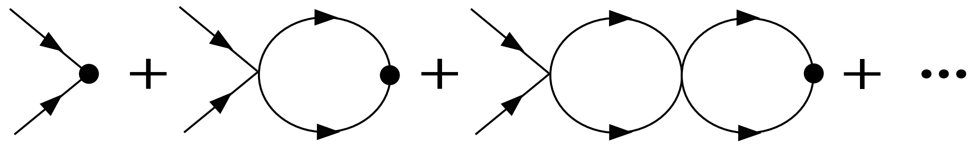

FIG. 1: Feynman diagrams contributing to the matrix element in Eq. (41). The black blob is the operator $\psi\psi$.

where d is the number of spatial dimensions and the factor $(\mu/2)^{3-d}$ is inserted to give the correct dimensions. We use DR and the PDS scheme [31] to evaluate the integral. The integral is linearly dependent on μ in the PDS scheme, reflecting the linear divergence, but μ independent in the MS scheme. Keeping the linear μ dependence is critical for finding the correct anomalous dimension for the composite operator $\psi\psi$.

The result for the matrix element is then

$$
\begin{aligned}
\langle 0|Z_{\psi\psi}(\mu)\psi\psi|\vec{p},-\vec{p}\rangle &= \frac{Z_{\psi\psi}(\mu)}{1+\frac{M}{4\pi}C_0(\mu))(\mu+ip)} \\
&= \frac{4\pi}{M}\left(\frac{Z_{\psi\psi}(\mu)}{C_0(\mu)}\right)\frac{1}{1/a+ip}\,,
\end{aligned}
\tag{45}
$$

where we have used Eq. (2). The matrix element is μ independent if $Z_{\psi\psi}(\mu)\propto C_0(\mu)$ and we can fix the constant of proportionality by demanding $\langle 0|Z_{\psi\psi}(\mu)\psi\psi|\vec{p},-\vec{p}\rangle = 1$ for $p^2=0$.[4] Then

$$
\begin{aligned}
Z_{\psi\psi}(\mu) &= \frac{M}{4\pi}\frac{C_0(\mu)}{a} \\
&= \frac{1}{1-\mu\, a}\,.
\end{aligned}
\tag{46}
$$

The anomalous dimension of the the operator $\psi\psi$ is then given by

$$
\begin{aligned}
\gamma_{\psi\psi} &= \mu\frac{d}{d\mu}\ln Z_{\psi\psi}(\mu) \\
&= \frac{\mu\, a}{1-\mu\, a}\,.
\end{aligned}
\tag{47}
$$

The effective field theory is a NRCFT when we take the limit $a\to\pm\infty$, and then $\gamma_{\psi\psi}=-1$. The scaling dimension of $\psi\psi$ is the naive dimension, $2\Delta_\psi=3$, plus the anomalous dimension, $\gamma_{\psi\psi}=-1$ so $\Delta_{\psi\psi}=2\Delta_\psi+\gamma_{\psi\psi}=2$, in agreement with the state-operator correspondence. Note that one must take the logarithmic derivative with respect to μ at finite a, then take the limit $a\to\pm\infty$. If the limit is taken prior to computing the derivative, then $Z_{\psi\psi}(\mu)=0$. However, this is an artifact of the boundary condition that $\langle 0|Z_{\psi\psi}(\mu)\psi\psi|\vec{p},-\vec{p}\rangle=1$, which is no longer possible when $a=\pm\infty$. If we start with Eq. (45), take the limit $a\to\pm\infty$ we should demand that residue of the $1/p$ pole be a μ-independent constant and we again obtain $\gamma_{\psi\psi}=-1$.

[4] The same normalization condition is obtained if one requires that the sum of all Feynman diagrams yields the same result when evaluated in the minimal subtraction (MS) scheme, in which case all loop graphs are finite and $C_0=4\pi a/M$.

Another way of obtaining the scaling dimension of a primary operator, $\mathcal{O}$, in NRCFT is to consider the two-point function:

$$\begin{aligned} G_{\mathcal{O}}(\vec{x},t) &= \langle 0|\mathcal{O}(\vec{x},t)\,\mathcal{O}^\dagger(\vec{0},0)|0\rangle \\ &\propto \Theta(t)\,t^{-\Delta_{\mathcal{O}}}\exp\left(-iN_{\mathcal{O}}\frac{\vec{x}^{\,2}}{t}\right), \end{aligned} \tag{48}$$

where the second line of Eq. (48) follows from scale and Galilean invariance [26]. Note we assume $N_{\mathcal{O}} > 0$. Fourier transforming this Green's function yields

$$\begin{aligned} \tilde{G}_{\mathcal{O}}(\vec{p},E) &= \int d^dx\,dt\,e^{-iEt+i\vec{p}\cdot\vec{x}}\,G_{\mathcal{O}}(\vec{x},t) \\ &\propto \frac{1}{(E-\frac{p^2}{2N_{\mathcal{O}}})^{d/2-\Delta_{\mathcal{O}}+1}} \end{aligned} \tag{49}$$

This formulae can be used once additive renormalizations are carried out. For example, in the noninteracting theory, for $d=3$,

$$\tilde{G}_{\psi\psi}(\vec{p},E) = -i\frac{M}{4\pi}\left(\mu-\sqrt{-ME+\frac{p^2}{4}+i\epsilon}\right) \tag{50}$$

After removing the μ dependence using an additive renormalization, we can compare with Eq. (49) to obtain $\Delta_{\psi\psi}=3$, which is the correct answer for a free theory. For the interacting theory,

$$\tilde{G}_{\mathcal{O}}(\vec{p},E) \;=\; \frac{M}{4\pi}\frac{i}{a^2}\left(\frac{1}{\mu-1/a}+\frac{1}{-\sqrt{-ME+p^2/4-i\epsilon}+1/a}\right). \tag{51}$$

In deriving this result we have included the factor $Z_{\psi\psi}(\mu)$ computed earlier. The first term can be removed by additive renormalization or else we can remove the cutoff dependence by taking $\mu\to\infty$. Then, the second term then yields $\Delta_{\psi\psi}=2$ when comparing with Eq. (49) in the limit $a\to\pm\infty$.

An alternative formulation of the NRCFT employs a composite field, which in the context of nuclear physics is called the dibaryon formalism [37]. This formalism is used most often in three-body calculations. In the dibaryon formalism one introduces a composite field, ϕ that has the same quantum numbers as $\psi\psi$ and removes the four-fermion interaction using a Hubbard-Stratonovich transformation [37]. In this formalism, Eq. (49) can be directly compared to the dibaryon propagator (see, e.g., Eq. (3) of Ref. [37]), in the limit $a\to\pm\infty$, $r_0\to 0$ and again one finds $\Delta_\phi=\Delta_{\psi\psi}=2$

It is also interesting to see how the scaling behavior of the two-body wavefunction dictates the anomalous dimension of the corresponding two-body operator in the field theory. This sheds further light on the relationship between the results of Ref. [27] and Ref. [26]. The unrenormalized sum of all graphs in Fig. 1 can be expressed quantum mechanically as

$$\langle 0|\psi\psi|\vec{p},-\vec{p}\rangle = \int\frac{d^dq}{(2\pi)^d}\,\langle\vec{q}\,|1+\frac{1}{E-H_0}\,T\,|\vec{p}\rangle\,, \tag{52}$$

where T is the transition operator that is a solution to the Lippmann-Schwinger equation,

$$T = V + V\frac{1}{E-H_0}\,T\,, \tag{53}$$

and matrix elements of V are $\langle\vec{q}|V|\vec{p}\rangle = C_0$. Here, we have let the number of spatial dimensions, d, be arbitrary. It well known from nonrelativistic quantum mechanics that the exact solution to the scattering wave equation with incoming particles with momentum $\vec{p}$ is

$$\chi_{\vec{p}}(\vec{x}) = \langle\vec{x}\,|1 + \frac{1}{E - H_0}\,T\,|\vec{p}\rangle \tag{54}$$

so we can write

$$\langle 0|\psi\psi|\vec{p},-\vec{p}\rangle = \chi_{\vec{p}}(0)\,, \tag{55}$$

so the matrix element can be interpreted as two-body wavefunction at the origin. However, this is divergent for the interacting theory. For two fermions at the unitary limit the two-body wavefunction, $\chi_{\vec{p}}(\vec{x})$, is proportional to r^{2-d} for small r. Inserting a complete set of state into Eq. (52) and regulating the expression with a hard cutoff in momentum space, we obtain

$$\langle 0|\psi\psi|\vec{p},-\vec{p}\rangle = \chi_{\vec{p}}(0) = \int^{\Lambda} \frac{d^d q}{(2\pi)^d} \int d^d x\, e^{-i\vec{q}\cdot\vec{x}}\, \chi_{\vec{p}}(\vec{x})\,. \tag{56}$$

This integral is of course divergent. The degree of divergence is determined by the $\vec{x} \to 0$ behavior of $\chi_{\vec{p}}(\vec{x})$ which is independent of $\vec{p}$. It is easily seen that the integral diverges like

$$\int^{\Lambda} \frac{d^d q}{(2\pi)^d} \int d^d x\, e^{-i\vec{q}\cdot\vec{x}} \frac{1}{r^{d-2}} \sim \Lambda^{d-2}\,. \tag{57}$$

If renormalize the matrix element in Eq. (52) with a multiplicative factor of $Z_{\psi\psi}(\Lambda)$, we must have $Z_{\psi\psi}(\Lambda) \propto \Lambda^{2-d}$ to get a finite answer for the matrix element. This leads to $\gamma_{\psi\psi} = 2-d$ which gives for the scaling dimension for $\Delta_{\psi\psi} = d + \gamma_{\psi\psi} = 2$, which is the correct answer for arbitrary d.

B. Three particles

The three-body problem in the presence of an external harmonic potential with infinite two-body scattering length was solved in Ref. [17]. For any interaction to take place two of the three particles must be in an S-wave. Ref. [17] solved the three-body problem for arbitrary l, where l is the total angular momentum of the three-body system, using the method of pseudopotentials. Since the interaction is modeled as zero-range, the three particles are free except when the coordinates of two of the particles coincide. The wavefunction is then a solution to the free Schrödinger equation subject to the boundary condition (for arbitrary a)

$$\lim_{r_{ij}\to 0} \psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) \propto \frac{1}{r_{ij}} - \frac{1}{a} + O(r_{ij}) \qquad (s_i = -s_j) \tag{58}$$

where $r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$, s_i and s_j are the spin quantum numbers of particle i and j, respectively, and the limit $r_{ij} \to 0$ is taken holding the coordinate of the third particle fixed. (For $s_i = s_j$ the wavefunction must vanish as $r_{ij} \to 0$.) In this paper, we will only consider the case of $l = 0$. It would be interesting to extend the analysis to arbitrary l but that is beyond the scope of this paper.

First we briefly review the solution obtained in Ref. [17]. Suppose we choose the spin states so that $s_1 = s_3 = -s_2$. The three-body wavefunction is parameterized as

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = (1 - P_{13})\, \psi_{\rm cm}(\vec{R}_{\rm cm})\, \frac{1}{r\,\rho}\, \chi(r, \rho)\,, \tag{59}$$

where $R_{\rm cm}$ is the center-of-mass coordinate, $r = |\vec{r}_1 - \vec{r}_2|$, $\rho = |2\,\vec{r}_3 - \vec{r}_1 - \vec{r}_2|/\sqrt{3}$, and the operator P_{13} interchanges $\vec{r}_1$ and $\vec{r}_3$. The wavefunction for the center-of-mass coordinate, $\psi_{\rm cm}(\vec{R}_{\rm cm})$, is a solution of the simple harmonic oscillator Hamiltonian, so $E_{\rm cm} = \omega(2n + l + 3/2)$. The function $\chi(r, \rho)$ obeys the following differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \rho^2} - \frac{M^2\omega^2}{4}(r^2 + \rho^2) + M(E - E_{\rm cm})\right) \chi(r, \rho) = 0\,. \tag{60}$$

Imposing the boundary conditions in Eq. (58), and demanding the wavefunction be finite as $\rho \to 0$, one finds that

$$\frac{\partial}{\partial r}\chi(0, \rho) + \frac{1}{a}\chi(0, \rho) - \frac{4}{\sqrt{3}\rho}\chi\left(\frac{\sqrt{3}}{2}\rho, \frac{1}{2}\rho\right) = 0\,, \qquad \chi(r, 0) = 0\,. \tag{61}$$

For $a = \pm\infty$, it is possible to solve the boundary condition with a factorized solution, $\chi(r, \rho) = F_n(R)\phi_n(\alpha)$, where $2\,R^2 = r^2 + \rho^2$, and $\alpha = \arctan(r/\rho)$. The function $\phi_n(\alpha)$ is determined by

$$\begin{aligned} -\frac{\partial^2}{\partial \alpha^2}\phi_n(\alpha) &= s_{0,n}^2\, \phi_n(\alpha)\,, \\ \phi_n(\pi/2) &= 0\,, \\ \phi_n'(0) &= \frac{4}{\sqrt{3}}\phi_n(\pi/3)\,. \end{aligned} \tag{62}$$

while $F(R)$ obeys the differential equation

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{s_{0,n}^2}{R^2} - M^2\omega^2 R^2 + 2M(E - E_{\rm cm})\right) F(R) = 0 \tag{63}$$

In addition we must have $\phi_n(0) \neq 0$, so that the residue of the $1/r_{12}$ pole in Eq. (58) is not equal to zero. The first two lines of Eq. (62) are solved by

$$\phi_n(\alpha) \propto \sin\left[\left(\alpha - \frac{\pi}{2}\right) s_{0,n}\right]. \tag{64}$$

while the third line of Eq. (62) leads to the transcendental equation for $s_{0,n}$

$$s_{0,n} \cos\left(\frac{\pi s_{0,n}}{2}\right) + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi s_{0,n}}{6}\right) = 0\,. \tag{65}$$

Note the solutions come in pairs, $s_{0,n} = \pm|s_{0,n}|$. $s_{0,n} = \pm 2$ is a solution to Eq. (65), however, inspection of Eq. (64) shows that for $s_{0,n} = \pm 2$, $\phi_n(0) = 0$, which will not satisfy the boundary condition in Eq. (58). There are no other integer solutions to Eq. (65), and all

remaining solutions to Eq. (65) give nontrivial solutions to the three body-problem. Numerical values of the five smallest values of $|s_{0,n}|$ are $2.16622, 5.12735, 7.11448, 8.83225, 11.06273$. The numbers $s_{0,n}$ determine the energy eigenvalues via Eq. (63). The solutions of Eq. (63) are [17]

$$F_n(R) \propto R^{s_{0,n}}\, e^{-R^2\, M\omega/2}\, L_q^{(s_{0,n})}(R^2\, M\,\omega)\,, \tag{66}$$

where $L_q^{(s_{0,n})}$ is a generalized Laguerre polynomial, and the energy eigenvalue is $E = E_{\rm cm} + \omega(s_{0,n} + 1 + 2q)$. Note that for the wavefunction to be square integrable, we must have $s_{0,n}$ positive in Eq. (66). The dependence on the quantum number q shows that for each $s_{0,n}$ there is an infinite tower of evenly spaced states whose energies are separated by $2\,\omega$, as expected from the $SL(2,R)$ algebra.

At this point we would like to demonstrate the correspondence between trapped eigenstates and zero-energy, scale-invariant eigenfunctions of the free Hamiltonian. To find these states, we can choose the same variables, $R_{\rm cm}, R$, and α, which were used to solve the trapped three-body problem. The function $\psi_{\rm cm}(\vec{R}_{\rm cm})$ is now a solution to the free particle Schrödinger equation, $\psi_{\rm cm}(\vec{R}_{\rm cm}) \propto e^{-i\vec{P}_{\rm cm}\cdot\vec{R}_{\rm cm}}$, and we should take $\vec{P}_{\rm cm} = 0$ to obtain a zero-energy state. The eigenvalue equations for $\phi_n(\alpha)$ are still Eq. (61), and $F_n(R)$ obeys Eq. (63) with $\omega = E = E_{\rm cm} = 0$. Thus, the solution for the zero-energy scale-invariant wavefunction has $F_n(R) \propto R^{s_{0,n}}$ and the zero-energy, scale-invariant solution to the three-body equation is

$$\begin{aligned}\psi_\nu(\vec{r}_1, \vec{r}_2, \vec{r}_3) &\propto (1 - P_{13})\,\frac{1}{r\,\rho} R^{s_{0,n}}\phi_n(\alpha) \\ &= (1 - P_{13})\, R^{s_{0,n}-2}\,\frac{\phi_n(\alpha)}{\sin(2\alpha)}\,. \end{aligned} \tag{67}$$

Clearly, the scaling exponent for this state is $\nu = s_{0,n} - 2$, so Eq. (28) tells us that the energy of the ground state of the infinite tower of states is $E = \omega(5/2 + s_{0,n})$, in agreement with the result obtained by direct solution of the three-body equations.

Now we would like to see how the effective field theory reproduces these results. We will show that the effective field theory allows one to derive a bound state equation which exhibits scaling solutions whose scaling exponents yield energy eigenvalues via the correspondence of Ref. [27]. Then we study how the state-operator correspondence can be used directly in three dimensions by calculating the anomalous dimension of an operator in the NRCFT and verifying that it reproduces the known result for the lowest energy state of three trapped particles in an S-wave.

In applications of effective field theory to three-body problems it has been found useful to employ the dibaryon formalism discussed earlier [33]. In the present context the composite field should be called a difermion, which we will denote ϕ, which has the same quantum numbers as $\psi\psi$. A Hubbard-Stratonovich transformation is used to trade the contact interaction in Eq. (1) for $\phi^\dagger\psi\psi$ and $\psi^\dagger\psi^\dagger\phi$ couplings. Loops of $\psi\psi$ contributing to the ϕ self-energies are summed to all orders to obtain the ϕ propagator. We refer readers to Ref. [33] for details on this procedure.

The scattering of ϕ and ψ proceeds via an infinite number of ladder-like diagrams. These can be resummed using a one-dimensional integral equation, which is pictured in Fig. 2. Double lines are ϕ propagators and single lines are ψ propagators. Evaluating the diagrams

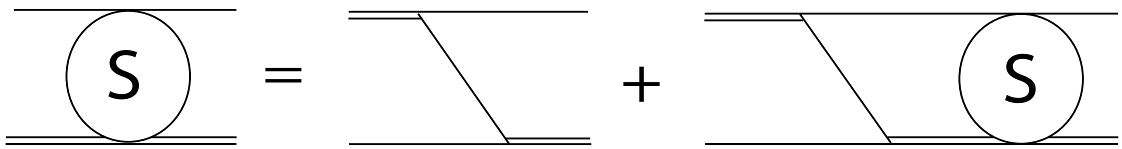

FIG. 2: Integral equation for $\phi\psi$ scattering.

in Fig. 2 and projecting onto the S-wave yields the half-off-shell integral equation

$$
\begin{aligned}
S(p,k) \;=\; & -\frac{M}{2pk}\log\left(\frac{p^2+pk+k^2-ME}{p^2-pk+k^2-ME}\right) \\
& -\frac{1}{\pi}\int_0^\infty dq\frac{1}{pq}\log\left(\frac{p^2+pq+q^2-ME}{p^2-pq+q^2-ME}\right)\frac{q^2}{-1/a+\sqrt{3q^2/4-ME}}\,S(q,k)\,.
\end{aligned}
\tag{68}
$$

Here the momentum k is on-shell, $3k^2/4 = ME + 1/a^2$, and the momenta p and q are off-shell. Note that $S(p,k)$ corresponds to the sum of ladder diagrams and does not include the LSZ factors required to obtain the on-shell amplitude when $p = k$, nor is it normalized to give the three-body scattering length for $p = k = 0$. Apart from these factors, the equation obtained here for $\phi\,\psi$ scattering is identical to that obtained in Ref. [33] for three nucleons in the $J = 3/2$ state of nucleons. Ref. [33] defines a half-off-shell amplitude, $a(p)$, which is normalized to the three-body scattering length, $a_3 = a(p = k)$. The function $S(p,k)$ is related to the function $a(p)$ of Ref. [33] by

$$
S(p,k) = -\frac{3M}{8}\frac{a(p)}{1/a+\sqrt{3p^2/4-ME}}\,. \tag{69}
$$

It is straightforward to reproduce the results of Ref. [17] using the NRCFT and the mapping of Ref. [27]. To find a zero-energy, scale-invariant eigenstate of the free space problem we can consider the equation for three-body bound states in the limit $a \to \pm\infty$ and $E = 0$. The bound state equation is obtained from Eq. (68) by dropping the inhomogeneous term in the integral equation. In the limit $E \to 0$ and $a \to \pm\infty$, the bound state equation becomes

$$
S(p,0) = -\frac{2}{\pi\sqrt{3}}\int_0^\infty \frac{dq}{p}\log\left(\frac{p^2+pq+q^2}{p^2-pq+q^2}\right)S(q,0)\,. \tag{70}
$$

Then one looks for solutions of the form $S(p,0) \propto p^{-s_{0,n}-1}$, which is possible if $s_{0,n}$ satisfies Eq. (65) [33]. The integral equation for $S(p,k)$ is finite and does not require renormalization. In order for the second diagram on the right hand side of the integral equation in Fig. 2 to converge for large q, $S(q,0)$ must vanish as $q \to \infty$, which then leads to $S(p,0) \propto p^{-|s_{0,n}|-1}$. To see how the $p \to \infty$ behavior of $S(p,0)$ is related to the scaling behavior of the many-body wavefunction, we recall the three-body position space wavefunctions can be obtained from $S(p,k)$ using the following transform [38]

$$
\chi(r,\rho) = \int_0^\infty dp\,S(p,k)\frac{p\sin\left(\sqrt{\frac{3}{4}}p\,\rho\right)}{-1/a+\sqrt{\frac{3}{4}p^2-ME}}e^{-r\sqrt{\frac{3}{4}p^2-ME}}\,. \tag{71}
$$

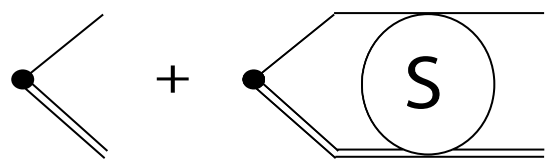

FIG. 3: Diagrams contributing to the renormalization of the operators $\mathcal{O}_1$.

It is straightforward to show that the function $\chi(r,\rho)$ obeys Eqs. (60,61). Eq. (60) and the second boundary condition in Eq. (61) follow directly from the definition in Eq. (71), while the first boundary condition in Eq. (61) can be obtained using the integral equation for $S(p,k)$. Taking the limit $E=0$, $a=\pm\infty$, and inserting the the asymptotic solution for $S(p,0)$, we find

$$\begin{aligned}\chi(r,\rho) &\propto \int_0^\infty dp\, p^{-s_{0,n}-1}\,\sin\left(\sqrt{\frac{3}{2}}\,p\,R\,\cos\alpha\right)\, e^{-\sqrt{3/2}\,p\,R\,\sin\alpha}\\ &\propto R^{s_{0,n}}\,\sin\left[s_{0,n}\left(\alpha-\frac{\pi}{2}\right)\right]\,. \end{aligned}\tag{72}$$

which is the correct form of the zero-energy, scale-invariant solution.

Finally we wish to understand the state-operator correspondence for the case of the three trapped fermions in an S-wave. As an example, we show how the state-operator correspondence can be used to calculate the lowest energy state of three harmonically trapped fermions in an S-wave. We compute the scaling dimension of the operator

$$\mathcal{O}_1=\phi\, i\frac{\overleftrightarrow{\partial}}{\partial t}\,\psi\,.\tag{73}$$

Operators that are a total time or space derivative are not primary, so $\mathcal{O}_1$ is the unique primary operator with one time derivative. S-wave operators with two space-derivatives can be put in the form $\phi\nabla^2\psi$ after integration by parts, and are therefore equivalent by the equations of motion for ψ. Therefore, in the noninteracting theory, where ϕ has dimension 3, $\mathcal{O}_1$ is the unique operator with naive dimension $13/2$. Note that the lowest energy state of three noninteracting fermions in an S-wave has energy $13/2\,\omega$, so the naive scaling dimension is consistent with the state-operator correspondence for the free theory. We compute the matrix element $\langle 0|Z_1(\Lambda)\,\mathcal{O}_1|\vec{p},-\vec{p}\rangle$. The diagrams that contribute to this matrix element are pictured in Fig. 3, which shows a tree-level graph and another graph which includes the half-off-shell amplitude $S(p,k)$. The off-shell legs of $S(p,k)$ are contracted with the operator $\mathcal{O}_1$ to form a loop. This graph sums all loop corrections to the matrix element $\langle 0|Z_1(\Lambda)\,\mathcal{O}_1|\vec{p},-\vec{p}\rangle$. The sum of all diagrams contributing to the renormalization of $\mathcal{O}_1$ in the limit $E, 1/a=0$ is given by

$$\begin{aligned}Z_1(\Lambda)\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p_0-\frac{p^2}{2M}+i\epsilon}\,\frac{4\pi}{M}\,\frac{-i}{\sqrt{Mp_0+p^2/4-i\epsilon}}\,2p_0\,iS(p,0)\\ = Z_1(\Lambda)\,\frac{4}{\sqrt{3}\,\pi M^2}\int_0^\Lambda dp\,p^3\,S(p,0)\,.\end{aligned}\tag{74}$$

We have included the factor $Z_1(\Lambda)$ for composite operator renormalization. A cutoff on the virtual loop momentum is used to regulate the loop integral. The asymptotic form of $S(p,0)$

is determined by Eq. (70), so in general we have

$$S(p,0) = \sum_m c_m p^{-|s_{0,m}|-1} \tag{75}$$

where the coefficients in the expansion, c_m, must be determined numerically from the solution to the full integral equation. Obviously the large p behavior is dominated by the smallest values of m in Eq. (75). For the operator $\mathcal{O}_1$, the only divergent contribution comes from the term $m = 1$, $|s_{0,1}| = 2.16622$. All other terms in Eq. (75) give a UV finite contribution to Eq. (74). The integral on the right hand side of Eq. (74) diverges as $\Lambda^{3-|s_{0,1}|}$, so

$$\gamma_1 = \Lambda \frac{d}{d\Lambda} Z_1(\Lambda) = |s_{0,1}| - 3\,. \tag{76}$$

The scaling dimensions of ϕ and ψ are $\Delta_\phi = 2$ and $\Delta_\psi = 3/2$, as discussed in the previous section, and the time derivatives add 2 to the naive dimension of $\mathcal{O}_1$. Adding the anomalous dimension, $\gamma_{\phi\psi}$, we find the scaling dimension, $\Delta_1 = 5/2 + |s_{0,1}|$, which, via the state-operator correspondence, is also in agreement with the result for the lowest energy state of three harmonically trapped fermions in an S-wave [17].

We should not consider the operator $\phi\psi$. The analog of this operator in the formulation of the theory without a difermion field would be $(\psi\psi)\psi = \epsilon^{\alpha\beta}\psi_\alpha\psi_\beta\psi$ which is not allowed because of Fermi statistics. A local operator which creates (or annihilates) three fermions at a point must have derivatives acting on at least one of the fermion fields. Therefore the operator $\phi\psi$ which seems allowed if ϕ is treated as a boson, must be excluded from consideration when classifying local operators in the NRCFT. This can also be seen from the state-operator correspondence for the noninteracting theory. The naive scaling dimension of $\phi\psi$ is 9/2 in this case, but there is no state of three trapped fermions with energy $9/2\,\omega$. So clearly one obtains a contradiction with the state-operator correspondence if $\phi\psi$ is allowed.

Note that the true ground state of three fermions at the unitary limit has $l = 1$. It would be interesting to derive the transcendental equations analogous to Eq. (65) for $l \neq 0$ from the integral equations for scattering in higher partial waves derived in Ref. [39]. These should give the numbers $s_{l,n}$ that determine the energy eigenvalues of three fermions in higher partial waves [17]. Another problem is to determine operators that correspond to states with energy eigenvalues $E = 5/2 + |s_{0,n}|$, $n \geq 2$. These come from operators with two or more time derivatives or four or more spatial derivatives, or mixed time and space derivatives. In the equations analogous to Eq. (74), these operators will lead to more factors of p_0 or p^2 which will make the integral more divergent. This leads to more terms in the sum in Eq. (75) contributing to the anomalous dimension. It should be possible to find a basis of operators in which the anomalous dimensions are given by the $s_{0,n}$ that are solutions to Eq. (65).

IV. CONCLUSIONS

We have studied the problem of fermions with infinite two-body scattering length confined in harmonic traps. The N-body problem can be mapped to problems involving N fermions in the absence of an external potential. One approach is to map solutions of the trapped problem to zero-energy, scale-invariant solutions to the Schrödinger equation in free space [17]. Another approach is to relate the energy levels of N-fermion states to the scaling

dimensions of primary operators in an NRCFT [26]. In this paper, we have shown that these two mappings are related by an automorphism of the $SL(2, R)$ conformal algebra of the NRCFT. This automorphism interchanges the internal Hamiltonian of the NRCFT with the harmonic trapping potential. This provides a simple, group theoretical way of deriving virial theorems for trapped Fermi gases at the unitary limit. The virial theorems apply for energy eigenstates as well as thermal ensembles and hold for both spin polarized and unpolarized gases.

One goal of this paper was to apply the state-operator correspondence [26] directly in the three spatial dimensions ($d = 3$), which is clearly the most important case. In Ref. [26], the state-operator correspondence was combined with ϵ expansions about $d = 2$ and $d = 4$ to do perturbative calculations of the energy levels. We sought to apply the state-operator correspondence directly in three dimensions using the NRCFT of Eq. (1). This is clearly more difficult because analytic results are only available for two fermions. For two fermions we showed how to use the state-operator correspondence to calculate the energy levels of two trapped fermions at the unitary limit. For three fermions, the NRCFT gives an integral equation for $\phi\psi$ scattering which can used to find the zero-energy, scale-invariant eigenfunctions which can be used to find the eigenfunctions of the three trapped fermions via the correspondence of Ref. [17]. We showed how to use the state-operator correspondence to calculate the energy of the lowest energy S-wave three-fermion trapped state. It would be interesting to extend application of the state-operator correspondence to all eigenstates of the trapped three fermion problem.

Since the problems of two and three trapped fermions in the unitarity limit can be solved using quantum mechanics and the pseudopotential boundary conditions of Eq. (58), an important question is whether the mappings of the trapped fermion problems to free space problems will be useful for obtaining new results. The virial theorems [27] are an example of results that the conformal symmetry of the NRCFT can provide in the absence of an exact solution of the quantum mechanics problem. It would be interesting if the integral equations of the effective field theory could be used to calculate corrections to energy levels from a finite scattering length, or obtain new results for problems with four or more fermions at the unitary limit. It would also be interesting if $SL(2, R)$ invariance can be used to obtain information about correlation functions of two-point functions of primary operators in the eigenstates of harmonically trapped fermions. For example, if $SL(2, R)$ invariance provides interesting constraints on correlation functions like Eqs. (48,49) with the vacuum replaced by the ground state of N trapped fermions, one could perhaps learn something about the low lying excitations of the ground state of a trapped gas of cold atoms at the unitary limit.

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APPENDIX A: TWO FERMION IS A HARMONIC TRAP IN EFT

In this appendix we solve the problem of two-particles interacting via short range forces in the presence of a harmonic potential. This problem was first solved in Ref. [9] and is typically analyzed using the method of pseudopotentials, see e.g. Refs. [10, 11, 12, 13, 14, 15, 16]. Here we solve it by evaluating the two-particle Green's function.

Consider the Green's function

$$G^{(0)}_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2) \;=\; \langle \vec{x}_3,\vec{x}_4|\frac{1}{E_{\rm tot}-H^{(0)}}|\vec{x}_1,\vec{x}_2\rangle\,, \tag{A1}$$

where $H(H^0)$ corresponds to the interacting (noninteracting) Hamiltonian. $G_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2)$ obeys the integral equation

$$\begin{aligned} G_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2) \;&=\; G^0_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2) \\ &\quad + C_0(\mu)\int d^Dy\, G^0_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{y},\vec{y}\,)\, G_{E_{\rm tot}}(\vec{y},\vec{y};\vec{x}_1,\vec{x}_2)\,. \end{aligned} \tag{A2}$$

This equation can be derived in quantum mechanics using a delta-function potential with coefficient $C_0(\mu)$ or from the Feynman diagrams of the field theory in Eq. (1) in position space. It is helpful to go to center of mass coordinates

$$\vec{x}_{1,2} = \vec{R} \pm \frac{1}{2}\vec{r} \qquad \vec{x}_{3,4} = \vec{R}\,' \pm \frac{1}{2}\vec{r}\,'\,, \tag{A3}$$

because the Hamiltonian factorizes in these coordinates. The noninteracting Green's function is given by

$$G^0_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2) = \sum_{\vec{n},\vec{m}} \frac{\psi_{\vec{m}}(\vec{R}\,')\,\phi^0_{\vec{n}}(\vec{r}\,')\,\psi_{\vec{m}}(\vec{R}\,)\,\phi^0_{\vec{n}}(\vec{r}\,)}{E_{\rm tot}-E(\vec{n})-E(\vec{m})}\,. \tag{A4}$$

Here $\vec{n}=(n_x,n_y,n_z)$ and $\vec{m}=(m_x,m_y,m_z)$. The $\psi_{\vec{m}}(\vec{R})$ are eigenfunctions of the $H_{\rm CM}$ with energy $E(\vec{m})$, and $\phi^0_{\vec{n}}(\vec{R})$ are eigenfunctions of the noninteracting $H_{\rm rel}$ with energy $E(\vec{m})$. The interacting Green's function $G^0_{E_{\rm tot}}$ has the same form as $G_{E_{\rm tot}}$ with $\phi^0_{\vec{n}}(\vec{R})$ replaced by eigenfunctions of $H_{\rm rel}$, $\phi_{\vec{n}}(\vec{R})$. Since $H_{\rm cm}$ is the same in either case, the $\psi_{\vec{m}}(\vec{R})$ are common to $G^0_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2)$ and $G_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2)$, we can project onto an energy eigenstate of $H_{\rm CM}$. Defining

$$G^{(0)}_E(\vec{r}\,',\vec{r}\,) \equiv \int d^d\vec{R}\,d^d\vec{R}\,'\,\psi_{\vec{m}}(\vec{R}\,)\,\psi_{\vec{m}}(\vec{R}\,')\,G^{(0)}_{E_{\rm tot}}(\vec{x}_3,\vec{x}_4;\vec{x}_1,\vec{x}_2)\,, \tag{A5}$$

where $E=E_{\rm tot}-E(\vec{m})$ and $E(\vec{m})$ is an eigenvalue of $H_{\rm cm}$, we find that $G_E(\vec{r}\,',\vec{r})$ obeys the equation

$$G_E(\vec{r}\,',\vec{r}\,) = G^0_E(\vec{r}\,',\vec{r}\,) + C_0(\mu)\,G^0_E(\vec{r}\,',\vec{0})\,G_E(\vec{0},\vec{r}\,)\,. \tag{A6}$$

The notation is similar to that used in Eq. (44), however here H_0 is the simple harmonic oscillator Hamiltonian. Setting $\vec{r}\,'=\vec{0}$ we obtain

$$G_E(\vec{0},\vec{r}\,) = \frac{G^0_E(\vec{0},\vec{r})}{1-C_0(\mu)\,G^0_E(\vec{0},\vec{0})}\,. \tag{A7}$$

The poles of this expression are solutions to

$$\frac{1}{C_0(\mu)} - G_E^0(\vec{0},\vec{0}) = 0\,, \tag{A8}$$

where $G_E^0(\vec{0},\vec{0}\,)$ is the Green's function for the simple harmonic oscillator, and is given by

$$\begin{aligned} G_E^0(\vec{0},\vec{0}\,) &= -\int_0^\infty dt\, \langle\vec{0}|e^{(E-H_0)t}|\vec{0}\,\rangle \\ &= \int_0^\infty dt\, e^{Et}\left(\frac{M\omega}{4\pi\sinh(\omega t)}\right)^{d/2} . \end{aligned} \tag{A9}$$

A transcendental equation similar to Eq. (A8) but with a different regulator was obtained in Ref. [40]. This integral is evaluated for negative $E - E_0$, where E_0 is the ground state of the oscillator, using dimensional regularization:

$$\begin{aligned} \int_0^\infty dt \frac{e^{-at}}{(\sinh t)^{d/2}} &= 2^{d/2-1}\int_0^1 du\, u^{a/2+d/4-1}\,(1-u)^{-d/2} \\ &= 2^{d/2-1}\frac{\Gamma[1-\frac{d}{2}]\Gamma[\frac{a}{2}+\frac{d}{4}]}{\Gamma[1-\frac{d}{4}+\frac{a}{2}]} . \end{aligned} \tag{A10}$$

Analytically continuing the integral from negative to positive E we find, for arbitrary d,

$$G_E^0(\vec{0},\vec{0}\,) = -\left(\frac{M}{4\pi}\right)^{d/2}(2\omega)^{d/2-1}\frac{\Gamma[1-\frac{d}{2}]\Gamma[-\frac{E}{2\omega}+\frac{d}{4}]}{\Gamma[1-\frac{d}{4}-\frac{E}{2\omega}]} . \tag{A11}$$

Just like the $G_E^0(\vec{0},\vec{0}\,)$ in the absence of the oscillator potential, this integral is linear divergent, but finite if evaluated using dimensional regularization. The integral is defined exactly as in the free space theory, multiplying the integral by $(\mu/2)^{3-d}$ and subtracting the pole at $d = 2$. We find

$$G_E^0(\vec{0},\vec{0}\,) = \frac{M}{4\pi}\left(-\mu + \sqrt{2M\omega}\,\frac{\Gamma[\frac{3}{4}-\frac{E}{2\omega}]}{\Gamma[\frac{1}{4}-\frac{E}{2\omega}]}\right) . \tag{A12}$$

Therefore we find the poles of the Green's function are located at,

$$\begin{aligned} 0 &= \frac{1}{C_0(\mu)} - G_E^0(\vec{0},\vec{0}) \\ &= \frac{M}{4\pi}\left(\frac{1}{a} - \sqrt{2M\omega}\,\frac{\Gamma[\frac{3}{4}-\frac{E}{2\omega}]}{\Gamma[\frac{1}{4}-\frac{E}{2\omega}]}\right) , \end{aligned} \tag{A13}$$

which is the transcendental equation first derived in Ref. [9]. This result is easily generalized to include effective range corrections. Effective range corrections and higher order terms in the effective range expansion can be incorporated using higher dimension operators with derivatives. We can choose a basis where each higher dimension operator contributes a

factor of $C_{2n}(ME)^n$ to the tree level scattering amplitude, see Ref. [41] for more details. The scattering amplitude is in the absence of an external potential is

$$\begin{aligned}\mathcal{A} &= \frac{-1}{\left(\sum_n C_{2n}(\mu)(ME)^n\right)^{-1} + \frac{M}{4\pi}(\mu + ip)} \\ &= \frac{4\pi}{M}\frac{1}{p\cot\delta(E) - ip}\end{aligned} \tag{A14}$$

Including the higher derivative operators in the Eq. (A6) for the Green's function, one finds the the formula in Eq. (A13) becomes

$$\begin{aligned}0 &= \frac{1}{\sum_n C_{2n}(\mu)(ME)^n} - G_E^0(\vec{0},\vec{0}\,) \\ &= \frac{M}{4\pi}\left(-p\cot\delta(E) - \sqrt{2m\omega}\,\frac{\Gamma[\frac{3}{4} - \frac{E}{2\omega}]}{\Gamma[\frac{1}{4} - \frac{E}{2\omega}]}\right) ,\end{aligned} \tag{A15}$$

In Ref. [10] it was pointed out that Eq. (A13) receives significant corrections when $a\sqrt{2M\omega} = a/a_{\rm osc} \geq 1$. Later, it was shown [11, 12, 13] showed that reliable results could be obtained by making the substitution

$$\frac{1}{a} \rightarrow -p\cot\delta(E)\,. \tag{A16}$$

This substitution was called the "effective-scattering length model", which we see here can be derived in a straightforward way using effective field theory.

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